Optimality Conditions of Vector Set-Valued Optimization Problem Involving Relative Interior

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Abstract

Firstly, a generalized weak convexlike set-valued map involving the relative interior is introduced in separated locally convex spaces. Secondly, a separation property is established. Finally, some optimality conditions, including the generalized Kuhn-Tucker condition and scalarization theorem, are obtained.

1. Preliminaries

Let X, Y and Z be three separated locally convex spaces, and let 0 denote the zero element for every space. Let K be a nonempty subset of Y. The generated cone of K is defined as $\mathrm{cone}(K)=\{\lambda a \mid a\in K, \lambda\geq 0\}$. A cone $K\subseteq Y$ is said to be **pointed** if $K\cap(-K)=\{0\}$. A cone $K\subseteq Y$ is said to be **nontrivial** if $K\neq\{0\}$ and $K\neq Y$.

Let Y^* and Z^* stand for the topological dual space of Y and Z, respectively. From now on, let C and D be nontrivial pointed closed-convex cones in Y and Z, respectively. The topological dual cone C^+ and strict topological dual cone C^{+i} of C are defined as

$$\begin{aligned} C^+ &= \{y^*\in Y^* \mid \langle y, y^*\rangle \geq 0, \forall y\in C\}, \\ C^{+i} &= \{y^*\in Y^* \mid \langle y, y^*\rangle > 0, \forall y\in C\setminus\{0\}\}, \end{aligned} \tag{1}$$

where $\langle y, y^*\rangle$ denotes the value of the linear continuous functional y^* at the point y. The meanings of D^+ and D^{+i} are similar.

Let K be a nonempty subset of Y. We denote by cl(K), int(K), and aff(K) the closed hull, topological interior, and affine hull of K, respectively.

Definition 1.1 Let K be a subset of Y. The **relative interior** of K is the set

$$\mathrm{ri}(K)=\{x\in K \mid \exists U, \text{a neighborhood of } x, \text{ such that } U\cap \mathrm{aff}(K)\subseteq K\}. \tag{2}$$

Now, we give some basic properties about the relative interior.

Lemma 1.2 Let K be a subset of Y. Let $k_0\in K$, $\bar{k}\in \mathrm{ri}(K), \alpha\in R$, and $\lambda\in(0,1]$. Then, (a) $\alpha\,\mathrm{ri}(K)=\mathrm{ri}(\alpha K)$;

(b) if K is convex, then $(1-\lambda)k_0+\lambda\bar{k}\in\mathrm{ri}(K)$. (3)

Proof (a) Since $\alpha\,\mathrm{aff}(K)=\mathrm{aff}(\alpha K)$, it is clear that $\alpha\,\mathrm{ri}(K)=\mathrm{ri}(\alpha K)$.

(b) Since $\bar{k}\in\mathrm{ri}(K)$, there exists V, a neighborhood of 0, such that

$$\left(\bar{k}+V\right)\cap\mathrm{aff}(K)\subseteq K. \tag{4}$$

By (4), we have

$$\left(\lambda\bar{k}+\lambda V\right)\cap(\lambda\,\mathrm{aff}(K))\subseteq\lambda K. \tag{5}$$

It follows from (5) that

$$\left((1-\lambda)k_0+\lambda\bar{k}+\lambda V\right)\cap\left((1-\lambda)k_0+\lambda\,\mathrm{aff}(K)\right)\subseteq(1-\lambda)k_0+\lambda K. \tag{6}$$

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It is clear that

$$(1-\lambda)k_0+\lambda \operatorname{aff}(K)=\operatorname{aff}(K). \tag{7}$$

Since K is convex, we have

$$(1-\lambda)k_0+\lambda K\subseteq K. \tag{8}$$

By (6), (7), and (8), we obtain

$$\left((1-\lambda)k_0+\lambda\bar{k}+\lambda V\right)\cap \operatorname{aff}(K)\subseteq K, \tag{9}$$

which implies that

$$(1-\lambda)k_0+\lambda\bar{k}\in \operatorname{ri}(K). \tag{10}$$

Remark 1.3 By Lemma 1.2, if K is a convex cone, then $\operatorname{ri}(K)\cup\{0\}$ is a convex cone.

Lemma 1.4 If K is a convex cone of Y, then

$$K+\operatorname{ri}(K)\subseteq \operatorname{ri}(K). \tag{11}$$

Proof If $\operatorname{ri}(K)=\varnothing$, it is clear that the conclusion holds. If $\operatorname{ri}(K)\neq\varnothing$, we have

$$K+\operatorname{ri}(K)=2\left(\frac{1}{2}K+\frac{1}{2}\operatorname{ri}(K)\right)\subseteq 2\operatorname{ri}(K)=\operatorname{ri}(2K)=\operatorname{ri}(K), \tag{12}$$

where Lemma 1.2(b) is used in the first inclusion relation and Lemma 1.2(a) is used in the second equality.

Lemma 1.5 Let $K\subseteq Y$ be a closed-convex set with $\operatorname{ri}(K)\neq\varnothing$. If $0\notin \operatorname{ri}(K)$, then there exists $y^*\in Y^*\setminus\{0\}$ such that $\langle k,y^*\rangle\geq 0$ for each $k\in K$.

2. Separation Property

From now on, we suppose that $\operatorname{ri}(C)\neq\varnothing$ and $\operatorname{ri}(D)\neq\varnothing$. Let A be a nonempty subset of X and $F:A\to 2^Y$ be a set-valued map on A. Write $F(A)=\bigcup_{x\in A}F(x)$.

Definition 2.1 Let A be a nonempty subset of X. A set valued map $F:A\to 2^Y$ is called **C-convexlike** on A if the set $F(A)+C$ is convex.

Definition 2.2 Let A be a nonempty subset of X. A set-valued map $F:A\to 2^Y$ is called **C-weak convexlike** on A if the set $F(A)+\operatorname{ri}(C)$ is convex.

Definition 2.3 Let A be a nonempty subset of X. A set-valued map $F:A\to 2^Y$ is called **generalized C-weak convexlike** on A if the set $\operatorname{cone}(F(A))+\operatorname{ri}(C)$ is convex.

Now, we consider the following two systems.

System 1: There exists $x_0\in A$ such that $F(x_0)\cap(-\operatorname{ri}(C))\neq\varnothing$.

System 2: There exists $y^*\in C^+\setminus\{0\}$ such that $\langle y,y^*\rangle\geq 0$, for all $y\in F(A)$.

Theorem 2.4 Let A be a nonempty subset of X.

(i)Suppose that $F: A \to 2^Y$ is generalized C-weak convexlike on A and $\mathrm{ri(cl(cone}F(A)+\mathrm{ri}(C)))=\mathrm{ri(cone}F(A)+\mathrm{ri}(C))\neq\varnothing.$ If System1 has no solution, then System 2 has solution.

(ii)If $y^* \in C^{+i}$ is a solution of System2, then System1 has no solution.

Proof (i)Firstly, we assert that $0 \notin \mathrm{cone}(F(A))+\mathrm{ri}(C).$ Otherwise, there exist $x_0 \in A, \alpha \geq 0$ such that $0 \in \alpha F(x_0))+\mathrm{ri}(C).$

Case 1. If $\alpha = 0,$ then $0 \in \mathrm{ri}(C).$ Thus, there exists U, a neighborhood of 0, such

$$U \cap \mathrm{aff}(C) \subseteq C. \tag{13}$$

Without loss of generality, we suppose that U is symmetric. It follows from (13) that

$$U \cap (-\mathrm{aff}(C)) \subseteq (-C). \tag{14}$$

It is clear that aff(C) is a linear subspace of Y. Therefore, $\mathrm{aff}(C) = -\mathrm{aff}(C).$ By (14), we have

$$U \cap \mathrm{aff}(C) \subseteq (-C). \tag{15}$$

By (13) and (15), we obtain

$$U \cap \mathrm{aff}(C) \subseteq C \cap (-C). \tag{16}$$

Since C is nontrivial, there exists $\overline{c} \in C \setminus \{0\}$. By the absorption of U, there exists $\lambda,$a sufficiently small positive number, such that

$$\lambda\overline{c} \in U \cap \mathrm{aff}(C) \subseteq C \cap (-C), \tag{17}$$

which contradicts that C is pointed.

Case 2. If $\alpha > 0,$there exists $y_0 \in F(x_0)$ such that $-y_0 \in \left(\frac{1}{\alpha}\right)\mathrm{ri}(C) \subseteq \mathrm{ri}(C),$ which contradicts $F(x) \cap (-\mathrm{ri}(C)) = \varnothing,$ for all $x \in A.$

Therefore, our assertion is true. Thus, we obtain

$$0 \notin \mathrm{ri(cl(cone}(F(A))+\mathrm{ri}(C))). \tag{18}$$

Since F is generalized C-weak convexlike on A, $\mathrm{cl(cone}(F(A))+\mathrm{ri}(C))$ is a closed-convex set. By Lemma 1.5, there exists $y^* \in Y^* \setminus \{0\}$ such that

$$\langle y, y^* \rangle \geq 0, \forall y \in \mathrm{cl(cone}(F(A))+\mathrm{ri}(C)). \tag{19}$$

So, $$\langle \alpha F(x)+c, y^* \rangle \geq 0, \forall x \in A, c \in \mathrm{ri}(C), \alpha \geq 0. \tag{20}$$

Letting $\alpha = 0$ in (20), we obtain

$$\langle c, y^* \rangle \geq 0, \forall c \in \mathrm{ri}(C). \tag{21}$$

We assert that $y^* \in C^+$. Otherwise, there exists $c' \in C$ such that $\langle c', y^* \rangle < 0,$ hence, $\langle \theta c', y^* \rangle < 0,$ for all $\theta > 0.$ By Lemma 1.4, we have

$$\theta c' + c \in \mathrm{ri}(C), \forall c \in \mathrm{ri}(C). \tag{22}$$

It follows from (21) that

$$\langle \theta c' + c, y^* \rangle \geq 0, \forall \theta > 0, c \in \mathrm{ri}(C). \tag{23}$$

Thus, we obtain

$$\theta\langle c', y^* \rangle + \langle c, y^* \rangle \geq 0, \forall \theta > 0, c \in \mathrm{ri}(C). \tag{24}$$

On the other hand, (24) does not hold when $\theta > -\dfrac{\langle c, y^* \rangle}{\langle c', y^* \rangle} \geq 0.$ Therefore, $\langle c, y^* \rangle \geq 0,$ for all $c \in C,$ that is, $y^* \in C^+.$

Letting $\alpha = 1$ in (20), we have

$$\left\langle F(x)+c, y^*\right\rangle \geq 0, \forall x \in A, c \in \mathrm{ri}(C). \tag{25}$$

Taking $c_0 \in \mathrm{ri}(C), \lambda_n > 0, \lim_{n\to\infty} \lambda_n = 0,$ we have

$$\left\langle F(x)+\lambda_n c_0, y^*\right\rangle \geq 0, \forall x \in A, n \in N. \tag{26}$$

Limiting (26), we obtain $\left\langle F(x), y^*\right\rangle \geq 0,$ for all $x \in A.$

(ii) Since $y^* \in C^{+i}$ is a solution of System 2, we have

$$\left\langle y, y^*\right\rangle \geq 0, \ \forall y \in F(A). \tag{27}$$

Now, we suppose that System1 has solution. Then, there exists $x_0 \in A$ such that $F(x_0)\cap(-\mathrm{ri}(C)) \neq \emptyset.$ Thus, there exists $y_0 \in F(x_0)$ such that $-y_0 \in \mathrm{ri}(C).$ It is clear that $-y_0 \neq 0.$ So, we have

$$\left\langle y_0, y^*\right\rangle < 0, \tag{28}$$

which contradicts (27).

3. Optimality Conditions

Let $F: A \to 2^Y$ and $G: A \to 2^Z$ be two set-valued maps from A to Y and Z, respectively. Now, we consider the following vector optimization problem of set-valued maps:

$$\begin{array}{ll} \min & F(x) \\ \text{subject to} & -G(x)\cap D \neq \emptyset. \end{array} \tag{VP}$$

The feasible set of (VP) is defined by

$$S = \{x \in A \mid -G(x)\cap D \neq \emptyset\}. \tag{29}$$

Now, we define

$$\begin{aligned} &W\,\mathrm{Min}(F(S), C) = \{y_0 \in F(S) \mid y_0 - y \notin \mathrm{ri}(C), \forall y \in F(S)\}, \\ &P\,\mathrm{Min}(F(S), C) = \{y_0 \in F(S) \mid (-C)\cap \mathrm{cl}(\mathrm{cone}(F(S)+C-y_0)) = \{0\}\}. \end{aligned} \tag{30}$$

Definition 3.1 A point x_0 is called a **weakly efficient solution** of (VP) if $x_0 \in S$ and $F(x_0)\cap W\,\mathrm{Min}(F(S), C) \neq \emptyset.$ A point pair (x_0, y_0) is called a **weak minimizer** of (VP) if $y_0 \in F(x_0)\cap W\,\mathrm{Min}(F(S), C).$

Definition 3.2 A point x_0 is called a **Benson properly efficient solution** of (VP) if $x_0 \in S$ and $F(x_0)\cap P\,\mathrm{Min}(F(S), C) \neq \emptyset.$ A point pair (x_0, y_0) is called a **Benson proper minimizer** of (VP) if $y_0 \in F(x_0)\cap P\,\mathrm{Min}(F(S), C).$

Let $I(x) = F(x)\times G(x),$ for all $x \in A.$ It is clear that I is a set-valued map from A to $Y\times Z,$ where $Y\times Z$ is a separated local convex space with nontrivial pointed closed-convex cone $C\times D.$ The topological dual space of $Y\times Z$ is $Y^*\times Z^*,$ and the topological dual cone of $C\times D$ is $C^+\times D^+.$

By Definition 2.3, we say that the set-valued map $I: A \to 2^{Y\times Z}$ is generalized $C\times D$-weak convexlike on A if $\mathrm{cone}I(A)+\mathrm{ri}(C\times D)$ is a convex set of $Y\times Z.$

Theorem 3.3 Let $\mathrm{ri}(\mathrm{cl}(\mathrm{cone}\, I^*(A)+\mathrm{ri}(C\times D)))=\mathrm{ri}(\mathrm{cone}\, I^*(A)+\mathrm{ri}(C\times D))\neq\varnothing.$ Suppose that the following conditions hold:
(i) (x_0, y_0) is a weak minimizer of (VP);
(ii) $I^*(x)$ is generalized $C\times D$-weak convexlike on A, where
$I^*(x)=(F(x)-y_0)\times G(x).$ Then, there exists $(y^*, z^*)\in C^+\times D^+$ with
$(y^*, z^*)\neq(0,0)$ such that

$$\begin{aligned}&\inf_{x\in A}\left(\left\langle F(x), y^*\right\rangle+\left\langle G(x), z^*\right\rangle\right)=\left\langle y_0, y^*\right\rangle,\\&\inf\left\langle G(x_0), z^*\right\rangle=0.\end{aligned}\tag{31}$$

Proof According to Definition 3.1, we have

$$\left(y_0-F(S)\right)\cap\mathrm{ri}(C)=\varnothing.\tag{32}$$

It is clear that $I^*(x)=I(x)-(y_0,0),$ for all $x\in A.$ We assert that

$$-I^*(x)\cap\mathrm{ri}(C\times D)=\varnothing, \forall x\in A.\tag{33}$$

Otherwise, there exists $\bar{x}\in A$ such that

$$-I^*(\bar{x})\cap\mathrm{ri}(C\times D)\neq\varnothing.\tag{34}$$

It is easy to check that $\mathrm{ri}(C\times D)=\mathrm{ri}(C)\times\mathrm{ri}(D).$

Therefore, $$-I^*(\bar{x})\cap(\mathrm{ri}(C)\times\mathrm{ri}(D))\neq\varnothing.\tag{35}$$

By (35), we obtain $$\left(y_0-F(\bar{x})\right)\cap\mathrm{ri}(C)\neq\varnothing,\tag{36}$$

$$-G(\bar{x})\cap\mathrm{ri}(D)\neq\varnothing.\tag{37}$$

It follows from (37) that $\bar{x}\in S.$ Thus, by (36), we have

$$\left(y_0-F(S)\right)\cap\mathrm{ri}(C)\neq\varnothing,\tag{38}$$

which contradicts (32). Therefore, (33) holds.
By Theorem 2.4, there exists $(y^*, z^*)\in C^+\times D^+$ with $(y^*, z^*)\neq(0,0)$ such that

$$\left\langle I^*(x),(y^*, z^*)\right\rangle\geq 0, \forall x\in A.\tag{39}$$

That is,

$$\left\langle F(x), y^*\right\rangle+\left\langle G(x), z^*\right\rangle\geq\left\langle y_0, y^*\right\rangle, \forall x\in A.\tag{40}$$

Since $x_0\in S,$ there exists $p\in G(x_0)$ such that $-p\in D.$

Because $z^*\in D^+,$ we obtain $\left\langle p, z^*\right\rangle\leq 0.$

On the other hand, taking $x=x_0$ in (40), we get

$$\left\langle y_0, y^*\right\rangle+\left\langle p, z^*\right\rangle\geq\left\langle y_0, y^*\right\rangle.\tag{41}$$

It follows that $\left\langle p, z^*\right\rangle\geq 0.$ So, $\left\langle p, z^*\right\rangle=0.$ Thus, we have

$$\left\langle y_0, y^*\right\rangle\in\left\langle F(x_0), y^*\right\rangle+\left\langle G(x_0), z^*\right\rangle.\tag{42}$$

Therefore, it follows from (40) and (42) that

$$\inf_{x\in A}\left(\left\langle F(x), y^*\right\rangle+\left\langle G(x), z^*\right\rangle\right)=\left\langle y_0, y^*\right\rangle.\tag{43}$$

Finally, taking again $x=x_0$ in (40), we obtain

$$\left\langle y_0, y^*\right\rangle+\left\langle G(x_0), z^*\right\rangle\geq\left\langle y_0, y^*\right\rangle.\tag{44}$$

So, $\left\langle G(x_0), z^* \right\rangle \geq 0.$ We have shown that there exists $p \in G(x_0)$ such that $\left\langle p, z^* \right\rangle = 0.$
Thus, we have

$$\inf \left\langle G(x_0), z^* \right\rangle = 0. \tag{45}$$

Theorem 3.4 Suppose that the following conditions hold:
(i) $x_0 \in S;$
(ii) there exist $y_0 \in F(x_0)$ and $\left(y^*, z^*\right) \in C^{+i} \times D^+$ such that

$$\inf_{x \in A} \left(\left\langle F(x), y^* \right\rangle + \left\langle G(x), z^* \right\rangle \right) \geq \left\langle y_0, y^* \right\rangle. \tag{46}$$

Then, x_0 is a weakly efficient solution of (VP).

Proof By condition (ii), we have

$$\left\langle F(x) - y_0, y^* \right\rangle + \left\langle G(x), z^* \right\rangle \geq 0, \forall x \in A. \tag{47}$$

Suppose to the contrary that x_0 is not a weakly efficient solution of (VP).
Then, there exists $x' \in S$ such that $\left(y_0 - F(x')\right) \cap \text{ri}(C) \neq \emptyset.$
Therefore, there exists $t \in F(x')$ such that $y_0 - t \in \text{ri}(C) \subseteq C \setminus \{0\}.$
Thus, we obtain

$$\left\langle t - y_0, y^* \right\rangle < 0. \tag{48}$$

Since $x' \in S,$ there exists $q \in G(x')$ such that $-q \in D.$
Hence, $\left\langle q, z^* \right\rangle \leq 0.$ (49)
Adding (48) to (49), we have

$$\left\langle t - y_0, y^* \right\rangle + \left\langle q, z^* \right\rangle < 0, \tag{50}$$

which contradicts (47). Therefore, x_0 is a weakly efficient solution of (VP).

The following example will be used to illustrate Theorem 3.4.

Example 3.5 Let $X = Y = Z = R^2,\ C = D = \left\{ \left(y_1, 0\right) \mid y_1 \geq 0 \right\},$ and $A = \left\{ \left(1,0\right)\left(1,2\right) \right\}.$ The set-valued map $F : A \to 2^Y$ is defined as follows:

$$\begin{aligned} F(1,0) &= \left\{ \left(y_1, y_2\right) \mid y_1 \geq 1, y_1 \leq y_2 \leq 2 \right\}, \\ F(1,2) &= \left\{ \left(y_1, y_2\right) \mid y_1 \leq 2, 1 \leq y_2 \leq y_1 \right\}. \end{aligned} \tag{51}$$

The set-valued map $G : A \to 2^Z$ is defined as follows:

$$\begin{aligned} G(1,0) &= \left\{ \left(y_1, y_2\right) \mid -1 \leq y_1 \leq 0, y_2 = 0 \right\}, \\ G(1,2) &= \left\{ \left(y_1, y_2\right) \mid -1 \leq y_1 \leq 0, 0 \leq y_2 \leq 1 \right\}. \end{aligned} \tag{52}$$

Let $x_0 = (1,0),\ y_0 = (1,1) \in F(x_0), \left\langle \left(y_1, y_2\right), y^* \right\rangle = y_1 + y_2,$ and $\left\langle \left(y_1, y_2\right), z^* \right\rangle = -y_1.$
It is clear that all conditions of Theorem 3.4 are satisfied.
Therefore, (1,0) is a weakly efficient solution of (VP).

Now, we consider the following scalar optimization problem $\left(VP\right)_\varphi$ of (VP):

$$\begin{aligned} &\min \quad \left\langle F(x), \varphi \right\rangle \\ &\text{subject to} \quad x \in S, \end{aligned} \tag{$(VP)_\varphi$}$$

where $\varphi \in Y^* \setminus \{0\}.$

Definition 3.6 If $x_0 \in S, y_0 \in F(x_0)$ and

$$\langle y_0, \varphi\rangle \le \langle y, \varphi\rangle, \forall y \in F(S), \tag{53}$$

then x_0 and (x_0, y_0) are called a **minimal solution** and a **minimizer** of $(VP)_{\varphi'}$ respectively.

Lemma 3.7 Let $U_1, U_2 \subset Y$ be two closed-convex cones such that $U_1 \cap U_2 = \{0\}$. If U_2 is pointed and locally compact, then $(-U_1^+) \cap U_2^+ \ne \varnothing$.

Lemma 3.8 If V is a subset of Y, then
(i) $cl(cone(V + ri(C))) = cl(coneV + ri(C))$,
(ii) $cl(cone(V + ri(C))) = cl(cone(V + C))$.
Proof (i) If $V = \varnothing$, it is obvious that

$$cl(cone(V + ri(C))) = cl(coneV + ri(C)). \tag{54}$$

If $V \ne \varnothing$, there exists $c \in ri(C)$. It is clear that

$$\lambda c \in coneV + ri(C), \forall \lambda \in (0, +\infty). \tag{55}$$

Letting $\lambda \to 0$ in (55), we have

$$0 \in cl(coneV + ri(C)). \tag{56}$$

Now, we will show that

$$cone(V + ri(C)) \subseteq (coneV + ri(C)) \cup \{0\}. \tag{57}$$

Let $y \in cone(V + ri(C))$.
Case 1. If $y = 0$, then $y \in (coneV + ri(C)) \cup \{0\}$.
Case 2. If $y \ne 0$, there exist $\alpha > 0, \upsilon \in V$, and $\bar{c} \in ri(C)$ such that

$$y = \alpha(\upsilon + \bar{c}) = \alpha\upsilon + \alpha\bar{c} \in coneV + ri(C) \subseteq (coneV + ri(C)) \cup \{0\}. \tag{58}$$

Therefore, (57) holds. Since Y is separated, by (56) and (57), we obtain

$$\begin{aligned} cl(cone(V + ri(C))) &\subseteq cl((coneV + ri(C)) \cup \{0\}) \\ &= cl(coneV + ri(C)) \cup cl\{0\} \\ &= cl(coneV + ri(C)) \cup \{0\} \\ &= cl(coneV + ri(C)). \end{aligned} \tag{59}$$

That is,

$$cl(cone(V + ri(C))) \subseteq cl(coneV + ri(C)). \tag{60}$$

Using the technique of Lemma 2.1 in [2], we easily obtain

$$coneV + ri(C) \subseteq cl(cone(V + ri(C))). \tag{61}$$

So,

$$cl(coneV + ri(C)) \subseteq cl(coneV + ri(C)). \tag{62}$$

By (60) and (62), we have $cl(cone(V + ri(C))) = cl(coneV + ri(C))$.
(ii) It is obvious that

$$cl(cone(V + ri(C))) \subseteq cl(cone(V + C)). \tag{63}$$

We will show that

$$cone(V + C) \subseteq cl(cone(V + ri(C))). \tag{64}$$

It is clear that (64) holds if $V = \varnothing$. Now, we suppose that $V \ne \varnothing$.
Let $y \in cone(V + C)$, then there exist $\lambda \ge 0, \upsilon \in V$, and $c \in C$ such that

$$y = \lambda(\upsilon + c). \tag{65}$$

Since $ri(C) \ne \varnothing$, there exists $c_0 \in ri(C)$. It follows from Lemma 1.4 that

$$\frac{\lambda}{\alpha} c_0 + y = \lambda\left(\frac{1}{\alpha} c_0 + c + \upsilon\right) \in cone(V + ri(C)), \forall \alpha > 0. \tag{66}$$

Letting $\alpha \to +\infty$ in (66), we have
$$y \in \mathrm{cl}(\mathrm{cone}(V + \mathrm{ri}(C))), \qquad (67)$$
which implies that (64) holds. By (64), we obtain
$$\mathrm{cl}(\mathrm{cone}(V + C)) \subseteq \mathrm{cl}(\mathrm{cone}(V + \mathrm{ri}(C))). \qquad (68)$$
By (63) and (68), we have
$$\mathrm{cl}(\mathrm{cone}(V + \mathrm{ri}(C))) = \mathrm{cl}(\mathrm{cone}(V + C)). \qquad (69)$$

Theorem 3.9 Suppose that the following conditions hold:
(i) $C \subseteq Y$ is locally compact;
(ii) (x_0, y_0) is a Benson proper minimizer of (VP);
(iii) $F - y_0$ is generalized C-weak convexlike on S.
Then, there exists $\varphi \in C^{+i}$ such that (x_0, y_0) is a minimizer of $(VP)_\varphi$.

Proof By condition (ii), we have
$$(-C) \cap \mathrm{cl}(\mathrm{cone}(F(S) + C - y_0)) = \{0\}. \qquad (70)$$
By Lemma 3.8 and condition (iii), we obtain that $\mathrm{cl}(\mathrm{cone}(F(S) + C - y_0))$ is a closed-convex cone.
Thus, condition of Lemma 3.7 are satisfied. Therefore, there exists $\varphi \in C^{+i}$ such that
$$\varphi \in (\mathrm{cl}(\mathrm{cone}(F(S) + C - y_0)))^+. \qquad (71)$$
Since $F(S) - y_0 \subseteq \mathrm{cl}(\mathrm{cone}(F(S) + C - y_0))$, we obtain
$$\langle y - y_0, \varphi \rangle \geq 0, \forall y \in F(S). \qquad (72)$$
That is, $$\langle y, \varphi \rangle \geq \langle y_0, \varphi \rangle, \forall y \in F(S). \qquad (73)$$
So, (x_0, y_0) is a minimizer of $(VP)_\varphi$.

Conclusion

In this paper, our results are very useful to form Lagrange multipliers rule and establish duality theory.

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